JOURNAL OF ALGEBRA **175**, 811–819 (1995)

Gorenstein Graded Rings Associated to Ideals of Analytic Deviation 2

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Communicated by Melvin Hochster

Received March 16, 1993

1. INTRODUCTION

Let A be a Noetherian local ring of dimension d. Let $\mathfrak{m}$ denote the maximal ideal in A and assume that the field $A/\mathfrak{m}$ is infinite. For each ideal $I\ (\neq A)$ in A, we put $R(I) = A[It]$ in $A[t]$ (here t is an indeterminate over A) and $G(I) = R(I)/IR(I)$. Let J be a minimal reduction of I whence $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n \geq 0$. We put $r_J(I) = \min\{n \geq 0 \mid I^{n+1} = JI^n\}$, which we call the reduction number of I with respect to J. Let $\lambda(I) = \dim A/\mathfrak{m} \otimes_A G(I)$ denote the analytic spread of I. Hence J is minimally generated by $\lambda(I)$ elements [NR]. Following [HH1], we define $\operatorname{ad} I = \lambda(I) - \operatorname{ht}_A I$ and call it the analytic deviation of I.

In this paper we are interested in the Gorensteinness of the Rees algebra $R(I)$ and the associated graded ring $G(I)$ for ideals I of analytic deviation 2. Our purpose is to prove the following

THEOREM (1.1). *Let I be an ideal in a Gorenstein local ring A with infinite residue class field. Assume that* (i) *A/I is a Cohen–Macaulay ring,* (ii) *$s = \operatorname{ht}_A I \geq 1$ and the ideal $IA_\mathfrak{p}$ is generated by an $A_\mathfrak{p}$-regular sequence of length s for all $\mathfrak{p} \in \mathfrak{F} := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq I$ and $\dim A_\mathfrak{p} \leq s + 1\}$, and* (iii) *$\operatorname{ad} I = 2$. Let J denote a minimal reduction of I. Then $G(I)$ is a Gorenstein ring if and only if $r_J(I) \leq 1$.*

* Both authors are partially supported by Grant-in-Aid for Cooperative Research.

0021-8693/95 $12.00

As a consequence of the theorem, we have

COROLLARY (1.2). *Let I and J be as in Theorem* (1.1). *Then $R(I)$ is a Gorenstein ring if and only if $s = 2$ and $r_J(I) \leq 1$.*

The theory of ideals having small analytic deviation started from the research [HH1, HH2]. In [HH1] Huckaba and Huneke studied certain ideals I of ad $I = 1$ or 2 in a Cohen–Macaulay or Gorenstein local ring A, and gave a practical criterion for these ideals to have symbolic powers equal to ordinary ones. Added to this, they explored in [HH2] the Cohen–Macaulayness of $R(I)$ and proved especially that $R(I)$ is a Cohen–Macaulay ring if $r_J(I) \leq 1$. The research of [HH2] was succeeded by [GH], in which the first author and Huckaba computed the a-invariant $a(G(I))$ of $G(I)$ (see [GW, (3.1.4)] for the definition of a-invariant) in terms of the reduction number $r_J(I)$ with respect to J, in the case where the ring $G(I)$ is Cohen–Macaulay and ad $I = 1$. Their computation [GH, 2.1 and 2.4] proved quite helpful in the authors' recent research [GN1] on the Gorensteinness of $G(I)$ for ideals I of analytic deviation 1. Since the assertions [GH, 2.1 and 2.4] have been successfully generalized by [GN2, (1.1) and (1.3)] to those on ideals of analytic deviation 2, naturally the next target is a criterion for the ring $G(I)$ to be Gorenstein, which we will give in our Theorem (1.1).

2. PRELIMINARY FOR THE CASE OF ANALYTIC DEVIATION 1

Let A be a Cohen–Macaulay local ring of dim $A = 1$. Let I be an ideal in A such that (i) A/I is a Cohen–Macaulay ring of dim $A/I = 1$, (ii) I is generically null, that is, $IA_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/I$, and (iii) ad $I = 1$. Let $J = cA$ be a minimal reduction of I. We put $\mathfrak{a} = (0) : I$, $\bar{A} = A/\mathfrak{a}$, and $\bar{I} = I\bar{A}$. Let $\mathbf{G} = G(I)$ and let $\varphi : \mathbf{G} \to G(\bar{I})$ denote the canonical epimorphism of the associated graded rings. Then by [GN1, (2.3)] we have an exact sequence

$$(2.1) \qquad 0 \to {}_{\rho}\mathfrak{a} \to \mathbf{G} \xrightarrow{\varphi} G(\bar{I}) \to 0$$

of graded $\mathbf{G}$-modules, where ${}_{\rho}\mathfrak{a}$ denotes $\mathfrak{a}$ which is considered to be, via the projection $\rho : \mathbf{G} \to \mathbf{G}_0 = A/I$, a graded $\mathbf{G}$-module concentrated in degree 0.

The purpose of this section is to prove the following (2.2), which we need in the next section to prove Theorem (1.1). The proof given below is a slight modification of those of [GH, Claims 2.5 and 2.9].

PROPOSITION (2.2). *Assume that A is a homomorphic image of a Gorenstein local ring and that $r_J(I) \leq 1$. Then the graded canonical module $K_{\mathbf{G}}$ of $\mathbf{G}$ is generated by elements of degree* 0.

Proof. Let $f = ct$. Then as f is $G(\bar{I})$-regular [GN1, (2.5)(1)], by (2.1) we have an isomorphism ${}_{\rho}\mathfrak{a} \cong [(0):_{\mathbf{G}} f]$ and an exact sequence

$$(2.3) \qquad 0 \to {}_{\rho}\mathfrak{a} \to \mathbf{G}/f\mathbf{G} \to G(\bar{I}/c\bar{A}) \to 0$$

of graded $\mathbf{G}$-modules (notice that $G(\bar{I})/fG(\bar{I}) \cong G(\bar{I}/c\bar{A})$; cf. [VV, 2.1]). Let us split the homothety $\mathbf{G}(-1) \xrightarrow{f} \mathbf{G}$ into two exact sequences

$$(2.4) \qquad 0 \to {}_{\rho}\mathfrak{a}(-1) \to \mathbf{G}(-1) \to f\mathbf{G} \to 0$$

$$(2.5) \qquad 0 \to f\mathbf{G} \to \mathbf{G} \to \mathbf{G}/f\mathbf{G} \to 0.$$

Let $\mathfrak{M} = \mathfrak{m}\mathbf{G} + \mathbf{G}_+$, the unique graded maximal ideal in $\mathbf{G}$, and apply the local cohomology functors $H^i_{\mathfrak{M}}(\cdot)$ ($i \in \mathbf{Z}$) to (2.4) and (2.5). Then as $\mathbf{G}$ is a Cohen–Macaulay ring of dimension 1 [GN1, (2.5)(1)], we have exact sequences

$$(2.6) \qquad 0 \to H^1_{\mathfrak{M}}({}_{\rho}\mathfrak{a})(-1) \to H^1_{\mathfrak{M}}(\mathbf{G})(-1) \xrightarrow{\varepsilon} H^1_{\mathfrak{M}}(f\mathbf{G}) \to 0$$

and

$$(2.7) \quad 0 \to H^0_{\mathfrak{M}}(\mathbf{G}/f\mathbf{G}) \xrightarrow{\tau} H^1_{\mathfrak{M}}(f\mathbf{G}) \xrightarrow{\pi} H^1_{\mathfrak{M}}(\mathbf{G}) \to H^1_{\mathfrak{M}}(\mathbf{G}/f\mathbf{G}) \to 0$$

of local cohomology modules, where the composite $\pi \circ \varepsilon$ is the homothety of $H^1_{\mathfrak{M}}(\mathbf{G})$ given by f. We will show that $[(0):_{H^1_{\mathfrak{M}}(\mathbf{G})} \mathfrak{M}] \subseteq [H^1_{\mathfrak{M}}(\mathbf{G})]_0$. Assume the contrary and choose $n \in \mathbf{Z}$ and $x \in [H^1_{\mathfrak{M}}(\mathbf{G})]_n$ so that $n \neq 0$, $x \neq 0$, and $\mathfrak{M}x = (0)$. Then as $fx = 0$, we have $\varepsilon(x) = \tau(y)$ for some $y \in [\mathbf{G}/(f\mathbf{G})]_{n+1}$. Notice that $\varepsilon(x) \neq 0$, because $H^1_{\mathfrak{M}}({}_{\rho}\mathfrak{a})(-1)$ is concentrated in degree 1 (cf. [GH, 2.2]); hence $y \neq 0$ and $\mathfrak{M}y = (0)$ too. On the other hand, by (2.3) we have $\mathbf{G}/f\mathbf{G}$ is concentrated within degrees 0 and 1, because so is the ring $G(\bar{I}/c\bar{A})$ (use the fact that $I^2 = JI$). Hence $n + 1 = 0$, so that $0 \neq y \in [\mathbf{G}/f\mathbf{G}]_0$ and $\mathfrak{m}y = (0)$. But, this is impossible, because $[\mathbf{G}/f\mathbf{G}]_0 \cong \mathbf{G}_0$ and $\mathbf{G}_0 = A/I$ is a Cohen–Macaulay ring of $\dim A/I = 1$. Thus $[(0):_{H^1_{\mathfrak{M}}(\mathbf{G})} \mathfrak{M}] \subseteq [H^1_{\mathfrak{M}}(\mathbf{G})]_0$ so that $H^1_{\mathfrak{M}}(\mathbf{G})$ is embedded in a direct sum of finite copies of the graded injective envelope $\mathbf{E}_{\mathbf{G}}(\mathbf{G}/\mathfrak{M})$ of $\mathbf{G}/\mathfrak{M}$. Hence $K_{\mathbf{G}}$ is generated by elements of degree 0 (cf. [HIO, (36.4)]).

3. PROOF OF THEOREM (1.1)

Let I be an ideal in a d-dimensional Gorenstein local ring A with infinite residue class field. We assume that our ideal I satisfies conditions (i), (ii), and (iii) stated in Theorem (1.1). Let J be a minimal reduction of I. Then by the proof of [HH2, 3.3]) we may choose a system $a_1, a_2, \ldots, a_s, b, c$ of generators for J so that the conditions in (3.1) below are fulfilled, where $K = (a_1, a_2, \ldots, a_s)$ and $L = K + bA$.

PROPOSITION (3.1). (1) *$a_1, a_2, \ldots, a_s$ is an A-regular sequence and b is A-regular.*

(2) *$IA_{\mathfrak{p}} = KA_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/I$.*

(3) *$a_1, a_2, \ldots, a_{s-1}$ forms part of a minimal system of generators of $IA_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{F}$.*

(4) *$K : b = K : I$ and $(K : I) \cap I = K$.*

(5) *$(L : c) \cap I = L$.*

(6) *$\mathrm{ht}_A(I + (K : I)) \geq s + 1$ and $\mathrm{ht}_A(I + (L : I)) \geq s + 2$.*

We put $\bar{A} = A/(K : I)$, $\bar{I} = I\bar{A}$, and $\bar{J} = J\bar{A}$. Then $\bar{A}$ is a Cohen–Macaulay ring of dimension $d - s$ [PS]. Let $K_{\bar{A}} = \mathrm{Hom}_A(\bar{A}, A/K)$, the canonical module of $\bar{A}$ [HK, 5.20]. Then, as $\mathrm{Hom}_A(\bar{A}, A/K) \cong (K : (K : I))/K = I/K$ and $\bar{I} = (I + (K : I))/(K : I) \cong I/K$ (cf. (3.1)(4)), we have that $\bar{I}$ is a canonical ideal of $\bar{A}$. Therefore, combining [HK, 6.13] with [GN2, (3.4)], we get the following

PROPOSITION (3.2). (1) *$K_{\bar{A}} = \bar{I}$, $\mathrm{ht}_{\bar{A}}\bar{I} = 1$, and $\bar{A}/\bar{I}$ is a Gorenstein ring of dimension $d - s - 1$.*

(2) *b is $\bar{A}$-regular, and $I\bar{A}_{\mathfrak{p}} = b\bar{A}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass}_{\bar{A}}\bar{A}/\bar{I}$.*

(3) *$\lambda(\bar{I}) = 2$ and $\bar{J}$ is a minimal reduction of $\bar{I}$. Hence $\mathrm{ad}\,\bar{I} = 1$.*

(4) *$r_{\bar{J}}(\bar{I}) = r_J(I)$, if $K \cap I^n = KI^{n-1}$ for all $n \in \mathbf{Z}$.*

Let $\mathbf{G} = G(I)$, $\mathbf{S} = G(I/K)$, and $\mathbf{T} = G(\bar{I})$. Let $\varphi : \mathbf{S} \to \mathbf{T}$ be the canonical epimorphism of the associated graded rings. Then as $K_{A/I} \cong (K : I)/K$, by [GN2, (3.6)] we have an exact sequence

$$(3.3) \qquad 0 \to {}_{\rho}(K_{A/I}) \to \mathbf{S} \xrightarrow{\varphi} \mathbf{T} \to 0$$

of graded $\mathbf{S}$-modules, where ${}_{\rho}(K_{A/I})$ denotes the canonical module $K_{A/I}$ of A/I which is considered to be, via the projection $\rho : \mathbf{S} \to \mathbf{S}_0 = A/I$, a graded $\mathbf{S}$-module concentrated in degree 0. Let $K_{\mathbf{S}}$ and $K_{\mathbf{T}}$, respectively, denoted the graded canonical modules of $\mathbf{S}$ and $\mathbf{T}$. Hence $K_{\mathbf{T}} = \mathrm{Hom}_{\mathbf{S}}(\mathbf{T}, K_{\mathbf{S}})$ and ${}_{\rho}(K_{A/I}) = \mathrm{Hom}_{\mathbf{S}}({}_{\rho}(A/I), K_{\mathbf{S}})$ [HK, 5.12].

We begin with the following proof of the *only if* part of Theorem (1.1).

PROPOSITION (3.4). *$r_J(I) \leq 1$, if $\mathbf{G}$ is a Gorenstein ring.*

Proof. Assume $\mathbf{G}$ is a Gorenstein ring and let $r = r_J(I)$. Then, as $\mathbf{G}$ is Cohen–Macaulay, by [GN2, (3.3)] $a_1t, a_2t, \ldots, a_st$ forms a $\mathbf{G}$-regular sequence, so that by [VV, 2.1 and 2.7] we have the equality $K \cap I^n = KI^{n-1}$ for all $n \in \mathbf{Z}$ and the canonical isomorphism $\mathbf{S} \cong \mathbf{G}/(a_1t, a_2t, \ldots, a_st)\mathbf{G}$ as well. Hence $\mathbf{S}$ is a Gorenstein ring with $a(\mathbf{S}) = a(\mathbf{G}) + s = r - 2$ (cf. [GW, (3.1.6)] and [GN2, (1.3)]). Thus $K_{\mathbf{S}} = \mathbf{S}(r-2)$. Take the $K_{\mathbf{S}}$-dual of (3.3). Then we have an exact sequence

$$0 \to K_{\mathbf{T}} \to \mathbf{S}(r-2) \xrightarrow{\sigma} {}_p(A/I) \tag{3.5}$$

of graded $\mathbf{S}$-modules, since $\mathrm{Hom}_{\mathbf{S}}({}_p(K_{A/I}), K_{\mathbf{S}}) = {}_p(A/I)$ by [HK, 6.1]. If $r \geq 3$, then considering the homogeneous components in (3.5) of degree $2 - r$ (< 0), we get an isomorphism $[K_{\mathbf{T}}]_{2-r} \cong A/I$ of A-modules. Therefore we have $(K : I)\cdot(A/I) = (0)$, so that $K : I = K$ by (3.1)(4). This is impossible, because $\mathrm{ht}_A K = \mathrm{ht}_A I$ ($= s$). If $r = 2$, then by (3.2)(4) we have $r_{\bar{J}}(\bar{I}) = 2$, too. On the other hand, as $\mathbf{G}$ is Cohen–Macaulay, we get depth $A/I^2 \geq d - s - 2$ by [GN2, (1.4)]. Hence depth $\bar{A}/\bar{I}^2 \geq d - s - 2$ by [GN2, (3.5)], so that $\mathbf{T}$ is a Cohen–Macaulay ring by (3.2) and [GN1, (1.5)]. Consequently, in the exact sequence (3.5) the homomorphism σ has to be an epimorphism (cf. [HK, 6.1]). Therefore, as $\mathbf{S}_0 = A/I$, we get $[K_{\mathbf{T}}]_0 = (0)$, which contradicts the fact $a(\mathbf{T}) = 0$ (cf. [GH, 2.4], recall that $r_{\bar{J}}(\bar{I}) = 2$). Thus $r_J(I) \leq 1$, if G is a Gorenstein ring.

LEMMA (3.6). *If $R(I)$ is a Gorenstein ring, then $s \geq 2$.*

Proof. Assume $s = 1$. Then we have $\mathrm{depth}_A I = d$, as A/I is a Cohen–Macaulay ring of dimension $d - 1$. Hence $\mathrm{Hom}_A(I, A) \cong A$ by [GNi, (4.2)], because $R(I)$ is a Gorenstein ring. Thus $I \cong A$ by [HK, 6.1], which is absurd.

We are now ready to prove Theorem (1.1).

Proof of Theorem (1.1). Assume that $r = r_J(I) \leq 1$. We will show $\mathbf{G}$ is a Gorenstein ring. First of all, recall that $\mathbf{G}$ is a Cohen–Macaulay ring (cf. [HH2, 4.1] or [GN2, (4.6)]). Therefore, by the reduction technique modulo super-regular sequences given in [GN2, (3.9) and (3.10)], we may assume without loss of generality that $s = 1$ and $d = 3$. Let $a = a_1$. Then $\mathbf{S}$ ($= \mathbf{G}/at\mathbf{G}$) and $\mathbf{T}$ are both Cohen–Macaulay rings of dimension 2 and, taking the $K_{\mathbf{S}}$-dual of (3.3), we have the exact sequence

$$0 \to K_{\mathbf{T}} \to K_{\mathbf{S}} \to {}_p(A/I) \to 0 \tag{3.7}$$

of graded $\mathbf{S}$-modules (cf. Proof of (3.4)). Recall that $a(\mathbf{S}) = 0$ and $a(\mathbf{T}) = -1$ [GN2, (1.3) and (3.10); GH, 2.4]. Then we get an isomorphism $[K_{\mathbf{S}}]_0 \cong A/I$. We will show that $K_{\mathbf{S}}$ is generated by elements of degree 0.

To do this, notice that bt is $\mathbf{T}$-regular (cf., e.g., [GN2, (3.8)]) and we have by [VV, 2.1] an identification $G(\bar{I}/b\bar{A}) = \mathbf{T}/bt\mathbf{T}$, whence by [GW, (2.2.10)] an isomorphism $K_{G(\bar{I}/b\bar{A})} \cong [K_{\mathbf{T}}/btK_{\mathbf{T}}](1)$. Therefore by (2.2) we have $K_{\mathbf{T}}$ to be generated by elements of degree 1 (the ideal $\bar{I}/b\bar{A}$ of $\bar{A}/b\bar{A}$ satisfies the standard assumptions (i), (ii), and (iii) in Sect. 2; cf. (3.2)). Hence by [HSV, (2.4)] we have $K_{\mathbf{T}} = \mathrm{gr}_{\bar{I}}(K_{\bar{A}})(-1)$, where $\mathrm{gr}_{\bar{I}}(K_{\bar{A}})$ denotes the graded module associated to the filtration $(\bar{I}^n K_{\bar{A}})_{n \in \mathbf{Z}}$ of $K_{\bar{A}}$. However, because $K_{\bar{A}} = \bar{I}$ by (3.2)(1), we get $\mathrm{gr}_{\bar{I}}(K_{\bar{A}})(-1) = \mathbf{T}_+$ by definition, so that we have an isomorphism $K_{\mathbf{T}} \cong \mathbf{S}_+$, thanks to the exact sequence (3.3). Thus $\mathbf{T} = \mathrm{Hom}_{\mathbf{S}}(\mathbf{S}_+, K_{\mathbf{S}})$ by [HK, 6.1]. Now consider the exact sequence $0 \to \mathbf{S}_+ \to \mathbf{S} \xrightarrow{p} A/I \to 0$ and take the $K_{\mathbf{S}}$-dual of it. Then we get an exact sequence $0 \to_p (K_{A/I}) \to K_{\mathbf{S}} \to \mathbf{T} \to 0$ of graded $\mathbf{S}$-modules, by which we know that $K_{\mathbf{S}}$ is generated by elements of degree 0. Because $[K_{\mathbf{S}}]_0 = A/I$ as we have seen in (3.7), this proves $K_{\mathbf{S}}$ is a cyclic $\mathbf{S}$-module, whence $\mathbf{S}$ is a Gorenstein ring. As $\mathbf{S} = \mathbf{G}/at\mathbf{G}$ and as at is $\mathbf{G}$-regular [GN2, (3.3)], we have $\mathbf{G}$ to be a Gorenstein ring. This completes the proof of Theorem (1.1).

Proof of Corollary (1.2). By (3.6) we may assume $s \geq 2$. Hence $R(I)$ is a Gorenstein ring if and only if $\mathbf{G}$ is a Gorenstein ring and $a(\mathbf{G}) = -2$ [I, 3.1]. Thus the assertion (1.2) directly follows from [GN2, (1.3)] and (1.1).

If all the symbolic powers $I^{(n)}$ of I equal the ordinary powers I^n, by [HH1, Proof of Theorem 3.5] we have $r_J(I) \leq 1$, so that by (1.1) we get the following

COROLLARY (3.8). *$\mathbf{G}$ is a Gorenstein ring if $I^{(n)} = I^n$ for all $n \geq 1$. In particular, $\mathbf{G}$ is a Gorenstein ring, if it is an integral domain.*

4. AN ESTIMATION OF THE NUMBER $\mu_A(I)$ OF GENERATORS FOR I IN THE CASE WHERE G IS A GORENSTEIN RING

As is in Section 3, let I be an ideal in a d-dimensional Gorenstein local ring A with infinite residue class field. We assume that the ideal I satisfies conditions (i), (ii), and (iii) stated in Theorem (1.1). Let J be a minimal reduction of I. In this section we shall maintain the same notation as in Section 3.

We begin with the following

LEMMA (4.1). *Suppose $r_J(I) \leq 1$. Then $(\mathfrak{m} I^{n+1} + KI^n) \cap bI^n = b(\mathfrak{m} I^n + KI^{n-1})$ for all $n \geq 0$.*

Proof. Induction on n. Choose $x \in I^n$ so that $bx \in \mathfrak{m}I^{n+1} + KI^n$. If $n = 0$, then $bx \in \mathfrak{m}I + K$; so we have $x \in \mathfrak{m}$, because $a_1, a_2, \ldots, a_s, b, c$ is part of a minimal basis of I. Assume $n \geq 1$. Then as $I^2 = JI$, we have $\mathfrak{m}I^{n+1} + KI^n = (b,c)\mathfrak{m}I^n + KI^n$. Write $bx = by + cz + v$ with $y, z \in \mathfrak{m}I^n$ and $v \in KI^n$. Then as $z \in (L : c) \cap I^n$, we have by (3.1)(5) and [GN2, (4.4)]) that $z \in LI^{n-1}$. Let $z = w + bu$ with $w \in KI^{n-1}$ and $u \in I^{n-1}$. Then as $bu \in bI^{n-1} \cap (\mathfrak{m}I^n + KI^{n-1})$, we have, by the hypothesis of induction on n, $bu \in b(\mathfrak{m}I^{n-1} + KI^{n-2})$. Hence $u \in \mathfrak{m}I^{n-1} + KI^{n-2}$, as b is A-regular (cf. (3.1)(1)). On the other hand, as $bx = by + c(w + bu) + v$, we have $x - y - cu \in (K : b) \cap I^n$. Therefore by (3.1)(4) and [GN2, (4.5)] we get $x - y - cu \in KI^{n-1}$. Thus $x \in \mathfrak{m}I^n + KI^{n-1}$ as claimed.

By (3.3) we have the isomorphism $A/\mathfrak{m} \otimes_A \varphi : A/\mathfrak{m} \otimes_A \mathbf{s} \to A/\mathfrak{m} \otimes_A \mathbf{T}$ of $A/\mathfrak{m}$-algebras (notice that $\operatorname{Ker} \varphi = (I + (K : I))/I \subseteq \mathfrak{m}/I$; cf. [GN2, Sect. 3]) whence $\dim \mathbf{S}/\mathfrak{m}\mathbf{S} = 2$, as $\lambda(\bar{I}) = 2$ by (3.2)(3). We furthermore have

PROPOSITION (4.2). *$\mathbf{S}/\mathfrak{m}\mathbf{S}$ is a Cohen–Macaulay ring, if $r_J(I) \leq 1$.*

Proof. We will show that bt, ct forms an $\mathbf{S}/\mathfrak{m}\mathbf{S}$-regular sequence. As $\mathbf{S}/\mathfrak{m}\mathbf{S} = \oplus_{n \geq 0} I^n/(\mathfrak{m}I^n + KI^{n-1})$, bt is $\mathbf{S}/\mathfrak{m}\mathbf{S}$-regular by (4.1). Notice that $\mathbf{S}/(\mathfrak{m}\mathbf{S} + bt\mathbf{S}) = \oplus_{n \geq 0} I^n/(\mathfrak{m}I^n + LI^{n-1})$, as bt is $\mathbf{T}$-regular (cf. [GN2, (3.8)] and [VV, 2.1]). Choose $x \in I^n$ so that $cx \in \mathfrak{m}I^{n+1} + LI^n$. If $n = 0$, then $cx \in \mathfrak{m}I + L$ whence $x \in \mathfrak{m}$. Suppose $n \geq 1$. Then as $I^2 = JI$, we have $cx \in c\mathfrak{m}I^n + LI^n$. Write $cx = cy + z$ with $y \in \mathfrak{m}I^n$ and $z \in LI^n$. Then $x - y \in (L : c) \cap I^n = LI^{n-1}$ (cf. (3.1)(5) and [GN2, (4.4)]); so we have $x \in \mathfrak{m}I^n + LI^{n-1}$. Thus ct is $\mathbf{S}/(\mathfrak{m}\mathbf{S} + bt\mathbf{S})$-regular.

Let us note the main result of this section.

THEOREM (4.3). *Suppose that $\mathbf{G}$ is a Gorenstein ring. Then $\mu_A(I) = s + 2 + \operatorname{length}_A(\operatorname{Ext}_A^{d-s-2}(A/\mathfrak{m}, A/I^2))$.*

Proof. If $d > s + 2$, choose the element x as is in [GN2, (3.9)]. Then as x is $\mathbf{G}$-regular, passing to the ideal IC in the ring $C = A/xA$, we can reduce $d - s$ by one. Hence without loss of generality we may assume $d = s + 2$. Let $\mathfrak{M} = [\mathbf{S}/\mathfrak{m}\mathbf{S}]_+$. Then as bt, ct is an $\mathbf{S}/\mathfrak{m}\mathbf{S}$-regular sequence (cf. (4.2)), we get an isomorphism between the socle of $H^2_{\mathfrak{M}}(\mathbf{S}/\mathfrak{m}\mathbf{S})$ and that of $[H^0_{\mathfrak{M}}(\mathbf{S}/(\mathfrak{m}\mathbf{S} + (bt, ct)\mathbf{S}))](2)$. Hence

$$
\begin{aligned}
\operatorname{length}_A\left(\left[K_{\mathbf{S}/\mathfrak{m}\mathbf{S}}\right]_1\right) &= \operatorname{length}_A\left([\mathbf{S}/(\mathfrak{m}\mathbf{S} + (bt, ct)\mathbf{S})]_1\right) \\
&= \operatorname{length}_A(I/(\mathfrak{m}I + J)) \\
&= \mu(I) - (s + 2).
\end{aligned}
$$

On the other hand, as $\mathbf{S} = \mathbf{G}/(a_1 t, a_2 t, \ldots, a_s t)\mathbf{G}$ is a Gorenstein ring, we have $K_{\mathbf{S}/\mathfrak{m}\mathbf{S}} = [(0) :_{\mathbf{S}} \mathfrak{m}]$ by [HK, 5.20], whence $[K_{\mathbf{S}/\mathfrak{m}\mathbf{S}}]_1 \cong \operatorname{Hom}_A(A/\mathfrak{m}, I/(I^2 + K))$. Because A/I is a Cohen–Macaulay ring of dimension 2, we get

$$\operatorname{Hom}_A(A/\mathfrak{m}, I/(I^2 + K)) \cong \operatorname{Hom}_A(A/\mathfrak{m}, A/(I^2 + K)).$$

On the other hand, the exact sequence

$$0 \to (I^2 + K)/I^2 \to A/I^2 \to A/(I^2 + K) \to 0$$

yields an isomorphism $\operatorname{Hom}_A(A/\mathfrak{m}, A/(I^2 + K)) \cong \operatorname{Hom}_A(A/\mathfrak{m}, A/I^2)$, since $(I^2 + K)/I^2 \cong K/(K \cap I^2) = K/KI$ by [GN2, (3.2)] and since $K/KI \cong (A/I)^s$. Thus we get the required equality $\mu_A(I) = s + 2 + \operatorname{length}_A([K_{\mathbf{S}/\mathfrak{m}\mathbf{S}}]_1) = s + 2 + \operatorname{length}_A(\operatorname{Hom}_A(A/\mathfrak{m}, A/I^2))$.

By [GN2, (3.3)], we have depth $A/I^2 \geq d - s - 2$ if $\mathbf{G}$ is a Cohen–Macaulay ring. Hence combining (4.3) with (1.1), we get the following

COROLLARY (4.4). *Suppose that $r_J(I) \leq 1$. Then $I = J$ if and only if* $\operatorname{depth}_A A/I^2 \geq \dim A/I - 1$.

We would like to close this paper with the following (fairly well-known) two examples of *prime* ideals $\mathfrak{p}$ satisfying the standard assumptions (i), (ii), and (iii) in Theorem (1.1) and whose associated graded rings $G(\mathfrak{p})$ are Gorenstein rings. Let k be an infinite field. First, let $R = k[X_{1j}, X_{2j} \mid j = 1,2,3,4]$ be a polynomial ring in 8 variables over k. Let $\mathfrak{P}$ denote the ideal of R generated by the maximal minors of the generic matrix

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{bmatrix}.$$

We put $\mathfrak{M} = (X_{1j}, X_{2j} \mid j = 1,2,3,4)R$. Let $A = R_{\mathfrak{M}}$ and $\mathfrak{p} = \mathfrak{P}A$. Then $\operatorname{ht}_A \mathfrak{p} = 3$ and $\lambda(\mathfrak{p}) = 5$. As $A/\mathfrak{p}$ is a (Cohen–Macaulay) isolated singularity, we have that $\mathfrak{p}A_{\mathfrak{q}}$ is a complete intersection in $A_{\mathfrak{q}}$ for any $q \in \operatorname{Spec} A \setminus (\mathfrak{M}A)$ such that $\mathfrak{q} \supseteq \mathfrak{p}$. $G(\mathfrak{p})$ is an integral domain whence it is a Gorenstein ring by (3.8). We have $r_J(\mathfrak{p}) = 1$ for any minimal reduction J of $\mathfrak{p}$. Second, let $R = k[X_{ij} \mid 1 \leq i \leq j \leq 5]$ be a polynomial ring in 10 variables over k and let $\mathfrak{P}$ be the ideal of R generated by the pfaffians of order 4 of the generic alternating matrix

$$\begin{bmatrix} 0 & X_{12} & X_{13} & X_{14} & X_{15} \\ -X_{12} & 0 & X_{23} & X_{24} & X_{25} \\ -X_{13} & -X_{23} & 0 & X_{34} & X_{35} \\ -X_{14}4 & -X_{24} & -X_{34} & 0 & X_{45} \\ -X_{15} & -X_{25} & -X_{35} & -X_{45} & 0 \end{bmatrix}.$$

We put $\mathfrak{M} = (X_{ij} \mid 1 \leq i < j \leq 5)R$. Let $A = R_{\mathfrak{M}}$ and $\mathfrak{p} = \mathfrak{P}A$. Then $\mathrm{ht}_A \mathfrak{p} = 3$ and $\lambda(\mathfrak{p}) = 5$. The local ring $A/\mathfrak{p}$ is a (Gorenstein) isolated singularity, so that $\mathfrak{p}A_{\mathfrak{q}}$ is a complete intersection in $A_{\mathfrak{q}}$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus \{\mathfrak{M}A\}$ such that $\mathfrak{q} \supseteq \mathfrak{p}$. Hence $G(\mathfrak{p})$ is a Gorenstein ring by (1.1), because $\mathfrak{p}$ is a minimal reduction for itself.

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